SOLUTION OF THE VORTEX PROBLEM IN A BOUNDED MASS OF HEAVY LIQUID

(RESHENIE ZADACHI O VIKHRE V OGRANICHENNOI MASSE TIAZHELOI ZHIDKOSTI)

PMM Vol.28, № 4, 1964, pp. 771-777

O.M. KISELEV
(Kazan')

(Received March 13, 1964)

Solution of jet flow problems in hydrodynamics with consideration of gravitational force is of practical and theoretical interest. During the last decade a number of approximate methods were created each of which is applicable, as a rule, only to a narrow class of problems. In case of large Froude numbers F (the effect of gravitational force is insignificant) the method which is based on expansion of desired functions in series of powers of $1/F$ may turn out to be sufficiently convenient.

The idea of such a method is contained in the work of Voronetz [1 and 2]. In these papers the author examines the problem of flow of a heavy liquid from an orifice in a vertical wall. The solution is limited to terms of the first order with respect to $1/F$. The procedure for finding further approximations is not refined. In the work of Gurevich and Pykhteev [3], which is based on the idea of Voronetz, the problem of flow of a heavy liquid from under a baffle is solved to a first approximation. The work of Kostychev [4] in which an analogous method is applied to the examination of somewhat different questions should also be mentioned. In this paper the effectiveness of the method of small parameter is demonstrated in the application to the vortex problem in a bounded mass of a heavy liquid. Convergence of obtained series is proven.

1. Statement of the problem and solution. Planar steady potential flow of a heavy incompressible fluid from a vortex in the finite region of the plane $z = x + iy$ (the y-axis is oriented vertically up) is examined.

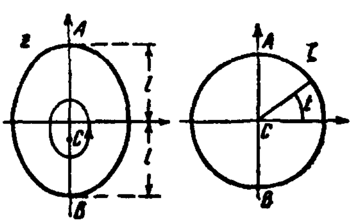

Fig. 1

The vortex is located at the point $C(z = z_0)$. At points A and B of the boundary the ordinate y reaches extreme values: $y_{max} = l$, and $y_{min} = -l$. The pressure along the boundary is constant, while, as will be shown below, the gravitational force which is acting on the fluid is equalized by an external concentrated force applied to the vortex.

From the Bernoulli equation it follows that the velocity V on the free surface must satisfy the relationships

$$V^2 = V_0^2\left(1 - \alpha\frac{y}{l}\right) \qquad \left(0 \leqslant \alpha = \frac{2gl}{V_0^2} < 1\right) \qquad (1.1)$$

Here g is the acceleration due to gravitational force, V_0 is the value of velocity at $y = 0$.

We will be looking for an analytical function $z = z(\zeta)$, representing a circle with a unit radius with the center at the origin of coordinates of the plane ζ over the region of flow in the z plane. It will be required that points $\zeta = i$, $-i$ and 0 will transform into points A, B and C correspondingly.

A complex flow potential $w = (\Gamma / 2\pi i) \ln \zeta$ is introduced; from (1.1) we obtain that for $\zeta = e^{it}$

$$\operatorname{Re}(\ln w' - \ln z') = \ln V_0 + {}^1/_2 \ln (1 - \alpha y / l) \tag{1.2}$$

Primes denote derivatives with respect to ζ.

The function $z(\zeta)$ and the magnitude of circulation Γ depend on α as a parameter. Assuming that in the vicinity of $\alpha = 0$ this dependence will be analytical we will represent the desired functions in the form of power series

$$z(\zeta, \alpha) = z_0(\zeta) + \alpha z_1(\zeta) + \alpha^2 z_2(\zeta) + \ldots, \qquad (z_k(e^{it}) = x_k(t) + iy_k(t)) \tag{1.3}$$

$$w(\zeta, \alpha) = \frac{\Gamma_0}{2\pi i}(1 + \alpha\gamma_1 + \alpha^2\gamma_2 + \ldots) \ln \zeta \tag{1.4}$$

Here Γ_0 and γ_k are real constants and $z_k(\zeta)$ are functions which are holomorphic within the circle.

Substitution of Expressions (1.3) and (1.4) into (1.2) gives

$$\operatorname{Re}\left[\ln\frac{\Gamma_0}{2\pi} + \ln(1 + \alpha\gamma_1 + \alpha^2\gamma_2 + \ldots) - \ln z_0' - \ln\left(1 + \alpha\frac{z_1'}{z_0'} + \alpha^2\frac{z_2'}{z_0'} + \ldots\right)\right] =$$

$$= \ln V_0 + \frac{1}{2}\ln\left[1 - \alpha\left(\frac{y_0}{l} + \alpha\frac{y_1}{l} + \alpha^2\frac{y_2}{l} + \ldots\right)\right] \qquad (\zeta = e^{it}) \tag{1.5}$$

Expanding logarithms which enter into Equation (1.5) in power series and then collecting coefficients for equal powers of α, we obtain an infinite series of conditions which must be satisfied by functions $z_k(\zeta)$ at the boundary of the circle.

$$\operatorname{Re}\left(\ln\frac{\Gamma_0}{2\pi} - \ln z_0'\right) = \ln V_0, \qquad \operatorname{Re}\left(\gamma_1 - \frac{z_1'}{z_0'}\right) = -\frac{1}{2}\frac{y_0}{l}$$

$$\operatorname{Re}\left(\gamma_2 - \frac{\gamma_1^2}{2} - \frac{z_2'}{z_0'} + \frac{1}{2}\frac{z_1'^2}{z_0'^2}\right) = -\frac{1}{2}\left(\frac{y_1}{l} + \frac{1}{2}\frac{y_0^2}{l^2}\right) \tag{1.6}$$

$$\operatorname{Re}\left(\gamma_3 - \gamma_1\gamma_2 + \frac{\gamma_1^3}{3} - \frac{z_3'}{z_0'} + \frac{z_1'z_2'}{z_0'^2} - \frac{1}{3}\frac{z_1'^3}{z_0'^3}\right) = -\frac{1}{2}\left(\frac{y_2}{l} + \frac{y_1y_2}{l^2} + \frac{1}{3}\frac{y_0^3}{l^3}\right)$$

$$\operatorname{Re}\left(\gamma_4 - \frac{\gamma_2^2}{2} - \gamma_1\gamma_3 + \gamma_1^2\gamma_2 - \frac{\gamma_1^4}{4} - \frac{z_4'}{z_0'} + \frac{1}{2}\frac{z_2'^2}{z_0'^2} + \frac{z_1'z_3'}{z_0'^2} - \frac{z_1'^2z_2'}{z_0'^3} + \frac{1}{4}\frac{z_1'^4}{z_0'^4}\right) =$$

$$= -\frac{1}{2}\left(\frac{y_3}{l} + \frac{1}{2}\frac{y_1^2}{l^2} + \frac{y_0y_2}{l^2} + \frac{y_0^2y_1}{l^3} + \frac{1}{4}\frac{y_0^4}{l^4}\right)$$

. . . .

In addition to this it is necessary to require that

$$x_0({}^1/_2\pi) = 0, \quad y_0({}^1/_2\pi) = -y_0(-{}^1/_2\pi) = l, \quad y_0^{\cdot}({}^1/_2\pi) = y_0^{\cdot}(-{}^1/_2\pi) = 0 \tag{1.7}$$

$$x_k({}^1/_2\pi) = y_k({}^1/_2\pi) = y_k(-{}^1/_2\pi) = y_k^{\cdot}({}^1/_2\pi) = y_k^{\cdot}(-{}^1/_2\pi) = 0 \qquad (k = 1, 2, \ldots) \tag{1.8}$$

(dots denote derivatives with respect to t)

From conditions (1.6) and (1.7) it follows that

$$z_0 = l\zeta, \qquad \Gamma_0 = 2\pi l V_0$$

Boundary conditions for the function $z_k(\zeta)$ take the form

$$\operatorname{Re}\left(\frac{z_k'}{l} - \gamma_k\right) = f_k(t) \qquad (\zeta = e^{it})$$

where the first part becomes known after determination of γ_i, $z_i(\zeta)$ $(i = 0, 1, \ldots, k-1)$. This circumstance gives the possibility to find functions $z_k(\zeta)$ successively one after the other. Constants γ_k together with three other real constants whivh arise in the process of determination of $z_k(\zeta)$ are found from Equations (1.8), one of which is fulfilled automatically.

It is not difficult to prove that for $m = 0,1,2,\ldots$

$$\gamma_{2m+1} = 0, \qquad z_{2m+1} = il \sum_{j=0}^{m+1} a_{2j}^{(2m+1)} \zeta^{2j}, \qquad z_{2m} = l \sum_{j=0}^{m} a_{2j+1}^{(2m)} \zeta^{2j+1} \tag{1.9}$$

Coefficients in Equation (1.9) are real.

With accuracy to terms α^4

$$z_{(4)} = l\,[\zeta - {}^1/_4\, i\alpha\,(1 + \zeta^2) - {}^1/_8\, \alpha^2\,(\zeta + \zeta^3) - {}^1/_{64}\, i\alpha^3\,(3 - 2\zeta^2 - 5\zeta^4) - \\ - {}^1/_{384}\, \alpha^4\,(17\zeta - 4\zeta^3 - 21\zeta^5)] \tag{1.10}$$

$$w_{(4)} = -ilV_0\,(1 - {}^1/_8\,\alpha^2 - {}^{11}/_{384}\,\alpha^4) \ln \zeta \tag{1.11}$$

(the order of approximation is given as the subscript in parenthesis). On the free surface we will have here

$$V_{(4)}^2 = V_0^2\,(1 - \alpha l^{-1}\, y_{(4)} + \alpha^5 \delta_{(4)}\,(t, \alpha))$$

Computations show that the function $\delta_{(4)}$ is weakly dependent on α and for $0 < \alpha \leqslant 0.5$ we have $|\delta_{(4)}| < 0.7$. With α increasing from 0 to 0.5 the contour of the free surface is slightly compressed from the sides, almost without losing symmetry with respect to x-axis, the circulation decreases and the vortex drops down. The location of the vortex is determined from (1.10) for $\zeta = 0$

$$z_c = -{}^1/_4\, i\alpha\,(1 + {}^3/_{16}\,\alpha^2)$$

It is also easy to convince oneself that for any approximation and for any α the contour of the free surface is symmetrical with respect to the y-axis, that the vortex is located on this axis and the function $\delta_{(n)}$ satisfies the condition $\delta_{(n)}(\pi + t) = \delta_{(n)}(-t)$.

2. Proof of convergence for solutions. We will prove that the series (1.3) and (1.4) constructed by us converge for $\zeta = e^{it}$, absolutely and uniformly with respect to the variable t at least for values of α with sufficiently small moduli. As support we will use here the work of Kantorovich [5] from which we borrow the following lemma.

L e m m a . If for two power series

$$a_1\xi + a_2\xi^2 + \ldots + a_n\xi^n + \ldots, \quad b_1\xi + b_2\xi^2 + \ldots + b_n\xi^n + \ldots$$

the coefficients satisfy the inequalities

$$|a_n| \leqslant \frac{A}{(n+1)^\beta}, \qquad |b_n| \leqslant \frac{B}{(n+1)^\beta} \qquad (n = 1, 2, \ldots;\ \beta > 1)$$

then, coefficients c_n of the power series representing their product satisfy the inequalities

$$|c_n| \leqslant \frac{AB}{(n+1)^\beta} M_\beta$$

where M_β is a constant only depending on β .

C o n s e q u e n c e . Applying the statement written above in turn to the powers of the function $\tau(\xi) = a_1\xi^n + a_2\xi^2 + \ldots + a_n\xi^n + \ldots$ we obtain for coefficients d_n of the function $\tau^k(\xi) = d_1\xi + d_2\xi^2 + \ldots + d_n\xi^n + \ldots$,

the following relationship
$$|d_n| \leqslant \frac{A^k}{(n+1)^\beta} M_\beta^{k-1}$$

The function $\varphi(t)$ is expanded in a power series as follows
$$\varphi(t) = \sum_{n=0}^{\infty} (a_n \cos nt + b_n \sin nt)$$

It is agreed upon to call as the norm of function $\varphi(t)$ the sum of moduli of coefficients in the expansion
$$\|\varphi(t)\| = \sum_{n=0}^{\infty} (|a_n| + |b_n|)$$

It is easy to convince oneself that
$$\|\varphi_1(t) + \varphi_2(t)\| \leqslant \|\varphi_1(t)\| + \|\varphi_2(t)\|, \quad \|\varphi_1(t)\cdot\varphi_2(t)\| \leqslant \|\varphi_1(t)\|\cdot\|\varphi_2(t)\|, \quad \|\overline{\varphi(t)}\| = \|\varphi(t)\|$$

We propose that for $1 \leqslant k \leqslant n$ the following statement is correct
$$\left\|\frac{z_k(e^{it})}{l}\right\| \leqslant \frac{CR^{k-1}}{(k+1)^\beta}, \qquad |\gamma_k| \leqslant \frac{3}{7}\frac{CR^{k-1}}{(k+1)^\beta} \tag{2.1}$$

Here C is a constant selected in such a manner that the statement (2.1) applies for $k = 1$. We will show that the constant R can be determined in such a fashion that the inequalities (2.1) will be fulfilled for $k = n \geqslant 2$.

With the aid of Equations (1.9) and the condition $y_n(\frac{1}{2}\pi) = 0$ it is not difficult to prove that
$$\left\|\frac{y_n(t)}{l}\right\| \leqslant \frac{1}{2}\left\|\frac{z_n'(e^{it})}{l}\right\|, \qquad \left\|\frac{z_n(e^{it})}{l}\right\| \leqslant \frac{3}{4}\left\|\frac{z_n'(e^{it})}{l}\right\| \tag{2.2}$$
$$|\gamma_n| \leqslant \left\|\operatorname{Re}\frac{z_n'(e^{it})}{l} - \gamma_n\right\|, \qquad \left\|\frac{z_n' e^{(it)}}{l}\right\| \leqslant \frac{7}{3}\left\|\operatorname{Re}\frac{z_n'(e^{it})}{l} - \gamma_n\right\| \tag{2.3}$$

Equation (1.5) is expanded into a power series in α, and coefficients for α^n are separated out. We will have
$$\gamma_n - \operatorname{Re}\frac{z_n}{l} = \left\{\frac{1}{n!}\frac{d^n}{d\alpha^n}\sum_{k=2}^{n}\frac{(-1)^k}{k}(\alpha\gamma_1 + \alpha^2\gamma_2 + \ldots + \alpha^{n-1}\gamma_{n-1})^k\right\}_{\alpha=0} -$$
$$-\frac{1}{2}\left\{\frac{1}{n!}\frac{d^n}{d\alpha^n}\sum_{k=2}^{n}\frac{(-1)^k}{k}\left(\alpha\frac{z_1'}{l} + \alpha^2\frac{z_2'}{l} + \cdots + \alpha^{n-1}\frac{z_{n-1}'}{l}\right)^k\right\}_{\alpha=0} -$$
$$-\frac{1}{2}\left\{\frac{1}{n!}\frac{d^n}{d\alpha^n}\sum_{k=2}^{n}\frac{(-1)^k}{k}\left(\alpha\frac{\bar z_1'}{l} + \alpha^2\frac{\bar z_2'}{l} + \cdots + \alpha^{n-1}\frac{\bar z_{n-1}'}{l}\right)^k\right\}_{\alpha=0} -$$
$$-\frac{1}{2}\frac{y_{n-1}}{l} - \frac{1}{2}\left\{\frac{1}{n!}\frac{d^n}{d\alpha^n}\sum_{k=2}^{n}\frac{1}{k}\left(\alpha\frac{y_0}{l} + \alpha^2\frac{y_1}{l} + \cdots + \alpha^{n-1}\frac{y_{n-2}}{l}\right)^k\right\}_{\alpha=0}$$

where
$$z_j' = z_j'(e^{it}), \qquad \bar z_j' = \overline{z_j'(e^{it})}, \qquad y_j = y_j(t)$$

Utilizing (2.1) and (2.2) and properties of norms of periodic functions we obtain from the last relationship

$$\left\|\operatorname{Re}\frac{z_n'}{l}-\gamma_n\right\|\leqslant\left\{\frac{1}{n!}\frac{d^n}{d\alpha^n}\sum_{k=2}^{n}\frac{1}{k}\left(\frac{3}{7}\frac{C}{R}\right)^k\left[\frac{\alpha R}{2^\beta}+\frac{(\alpha R)^2}{3^\beta}+\cdots+\frac{(\alpha R)^{n-1}}{n^\beta}\right]^k\right\}_{\alpha=0}+$$
$$+\left\{\frac{1}{n!}\frac{d^n}{d\alpha^n}\sum_{k=2}^{n}\frac{1}{k}\left(\frac{C}{R}\right)^k\left[\frac{\alpha R}{2^\beta}+\frac{(\alpha R)^2}{3^\beta}+\cdots+\frac{(\alpha R)^{n-1}}{n^\beta}\right]^k\right\}_{\alpha=0}+$$
$$+\frac{1}{4}\frac{CR^{n-2}}{n^\beta}+\frac{1}{2}\left\{\frac{1}{n!}\frac{d^n}{d\alpha^n}\sum_{k=2}^{n}\frac{1}{k}\left[\alpha+\frac{\alpha^2 C}{2\cdot 2^\beta}+\frac{\alpha^3 CR}{2\cdot 3^\beta}+\cdots+\frac{\alpha^{n-1}CR^{n-3}}{2\cdot(n-1)^\beta}\right]^k\right\}_{\alpha=0}$$

On the basis of results from Lemma

$$\left\{\frac{1}{n!}\frac{d^n}{d\alpha^n}\left[\frac{\alpha R}{2^\beta}+\frac{(\alpha R)^2}{3^\beta}+\cdots+\frac{(\alpha R)^{n-1}}{n^\beta}\right]^k\right\}_{\alpha=0}=$$
$$=R^n\left\{\frac{1}{n!}\frac{d^n}{(d\alpha R)^n}\left[\frac{\alpha R}{2^\beta}+\frac{(\alpha R)^2}{3^\beta}+\cdots+\frac{(\alpha R)^{n-k}}{n^\beta}\right]^k\right\}_{\alpha=0}\leqslant R^n\frac{M_\beta^{k-1}}{(n+1)^\beta}$$

We assume that the following inequalities are correct

$$\frac{C}{2^\beta}\geqslant 1,\qquad 2R\left(\frac{2}{3}\right)^\beta\geqslant 1 \tag{2.4}$$

Then

$$\left\{\frac{1}{n!}\frac{d^n}{d\alpha^n}\left[\alpha+\frac{\alpha^2 C}{2\cdot 2^\beta}+\frac{\alpha^3 CR}{2\cdot 3^\beta}+\cdots+\frac{\alpha^{n-1}CR^{n-3}}{2\cdot(n-1)^\beta}\right]^k\right\}_{\alpha=0}\leqslant$$
$$\leqslant\left\{\frac{1}{n!}\frac{d^n}{d\alpha^n}\left[\frac{\alpha C}{2^\beta}+\frac{\alpha^2 CR}{3^\beta}+\cdots+\frac{\alpha^{n-1}CR^{n-2}}{n^\beta}\right]^k\right\}_{\alpha=0}\leqslant R^n\left(\frac{C}{R}\right)^k\frac{M_\beta^{k-1}}{(n+1)^\beta}$$

In that way

$$\left\|\operatorname{Re}\frac{z_n'}{l}-\gamma_n\right\|\leqslant\frac{R^n}{M_\beta(n+1)^\beta}\sum_{k=2}^{\infty}\frac{1}{k}\left(\frac{3}{7}\frac{CM_\beta}{R}\right)^k+$$
$$+\frac{3R^n}{2M_\beta(n+1)^\beta}\sum_{k=2}^{\infty}\frac{1}{k}\left(\frac{CM_\beta}{R}\right)^k+\frac{1}{4}\frac{CR^{n-2}}{n^\beta}=$$
$$=\frac{R^n}{M_\beta(n+1)^\beta}\left[-\ln\left(1-\frac{3}{7}\frac{CM_\beta}{R}\right)-\frac{3}{7}\frac{CM_\beta}{R}\right]+$$
$$+\frac{3}{2}\frac{R^n}{M_\beta(n+1)^\beta}\left[-\ln\left(1-\frac{CM_\beta}{R}\right)-\frac{CM_\beta}{R}\right]+\frac{1}{4}\frac{CR^{n-2}}{n^\beta} \tag{2.5}$$

For the statement (2.1) to be correct at $k=n\geqslant 2$, taking into consideration (2.3) and (2.5), it is sufficient to require that the following inequality be satisfied

$$\frac{1}{4}\left(\frac{3}{2}\right)^\beta\frac{C}{R^2}+\frac{1}{M_\beta}\left[-\ln\left(1-\frac{3}{7}\frac{CM_\beta}{R}\right)-\frac{3}{7}\frac{CM_\beta}{R}\right]+$$
$$+\frac{3}{2M_\beta}\left[-\ln\left(1-\frac{CM_\beta}{R}\right)-\frac{CM_\beta}{R}\right]\leqslant\frac{3}{7}\frac{C}{R} \tag{2.6}$$

Let us assume $\beta=2$, then from [5] we will have $M_2=1.520$. Since $\gamma_1=0$ and $\|z_1'/l\|=1$, then to satisfy the first inequalities (2.4) and the inequalities (2.1) at $k=1$ it is sufficient to take $C=4$. Relationship (2.6) assumes the form

$$\frac{2.250}{R^2} + 0.658\left[-\ln\left(1 - \frac{2.606}{R}\right) - \frac{2.606}{R}\right] + $$
$$+ 0.987\left[-\ln\left(1 - \frac{6.080}{R}\right) - \frac{6.080}{R}\right] \leqslant \frac{1.714}{R}$$

From this $R \geqslant R_0 = 17.0$ (here the second of inequalities (2.4) will be satisfied). Therefore, the statement (2.1) is correct for any k at $\beta = 2$ and $R \geqslant R_0$. Consequently, the series

$$\Gamma = \Gamma_0 (1 + \alpha\gamma_1 + \alpha^2\gamma_2 + \ldots)$$
$$z'(e^{it}, \alpha) = z_0'(e^{it}) + \alpha z_1'(e^{it}) + \alpha^2 z_2'(e^{it}) + \ldots$$

will converge for $\alpha \leqslant 0.058 < 1/R_0$ (actually the limit of convergenece will be wider of course). With the aid of the second inequality of (2.2) it is easy to extend the last proof also to series (1.3).

3. Force acting on the vortex. Let us examine the system of external forces acting on a cylindrical volume of liquid bounded by a free surface and two planes normal to this surface. The distance between the two planes is equal to unity. The resultant of the gravitational force $P = -i\rho g S$ (where ρ is the density of the liquid and S is the base area of the liquid volume) and is directed along the y-axis . We will show that the force P is equalized by an external concentrated force acting on the vortex.

In fact, the following external force acts on the vortex

$$F = \frac{i\rho}{2}\int \overline{\left(\frac{dw}{dz}\right)^2}\, dz, \qquad \text{or} \qquad F = \frac{i\rho}{2}\int_L V^2 (dx + i\,dy)$$

Here in the first expression the integration is carried out along any arbitrary closed contour which lies in the plane of motion and encompasses the vortex. The second expression is applicable if L , the projection of free surface on the xy plane, is taken as the contour of integration. But on L condition (1.1) is satisfied, therefore

$$F = \frac{i\rho}{2} V_0^2 \int_L (dx + i\,dy) - \frac{i\rho}{2} V_0^2 \frac{\alpha}{l}\int_L (y\,dx + iy\,dy)$$

Taking into consideration that

$$\int_L dx = \int_L dy = \int_L y\,dy = 0, \qquad \int_L x\,dy = -\int_L y\,dx = S$$

we will have

$$F = i\rho g S \tag{3.1}$$

which was to be proved. Apparently Equation (3.1) is also applicable to any multipole in the bounded mass of liquid.

Assume the problem under examination to be solved to the nth approximation, i.e. functions z and w are found with accuracy to terms of the order α^n.

Then along L

$$V_{(n)}^2 = V_0^2\left(1 - \alpha\frac{y_{(n)}}{l} + \alpha^{n+1}\delta_{(n)}\right)$$

From (3.1) we obtain

$$F_{(n)} = i\rho g S_{(n)} + i\rho g l \alpha^n \int_L (\delta_{(n)}\, dx_{(n)} + i\delta_{(n)}\, dy_{(n)})$$

Taking into consideration equalities

$$\delta_{(n)}(\pi + t) = \delta_{(n)}(-t), \quad y_{(n)}^{\cdot}(\pi + t) = -y_{(n)}^{\cdot}(-t),$$

it is easy to prove that

$$\int_L \delta_{(n)}\, dy_{(n)} = 0$$

Therefore,

$$F_{(n)} = i\rho g\,(S_{(n)} + \alpha^n \sigma_{(n)}), \qquad \sigma_{(n)} = l \int_L \delta_{(n)}\, dx_{(n)}$$

at the same time when $P_{(n)} = -\,i\rho g S_{(n)}$. Thus the sum $P_{(n)}+F_{(n)}$ has the order α^2.

The last circumstance is conveniently used for checking the solution, applying Equation

$$F = i\rho\Gamma\,\overline{C}_1$$

here C_1 is the coefficient for $z - z_0$ in the expansion of the function w in the vicinity of point z_0.

4. **One approximate method.** In the solution of jet flow problems of general type by the method of expansion in powers of α the determination of even the first approximation presents as a rule considerable difficulties which are connected with the necessity of computing singular integrals depending on a number of parameters [3]. In the case where the solid boundary consists of vertical rectilinear sections it may turn out to be more advantageous to use another approximate method which permits to obtain the solution in the first approximation. The idea of this method consists of the following.

According to Equation (1.1) on the free surface

$$\ln V = \ln V_0 + {}^1/_2 \ln (1 - \alpha y / l) \tag{4.1}$$

Here l is some characteristic dimension. Considering α to be small and y/l bounded on the free surface we write

$$\ln V = \ln V_0 - {}^1/_2\, \alpha y / l$$

i.e. we will neglect on the right-hand side of Equation (4.1) terms of the order α^2 and higher.

We introduce the following analytical function (4.2)

$$F = \ln V_0 + \frac{i\alpha}{2l} z - \ln \frac{dw}{dz} \qquad \left(\mathrm{Re}\, F = \ln V_0 - \frac{\alpha}{2}\frac{y}{l} - \ln V,\ \mathrm{Im}\, F = \frac{\alpha}{2}\frac{x}{l} + \theta\right)$$

Here θ is the angle of inclination of velocity vector to the x-axis. On vertical rectilinear sections of a solid boundary $\mathrm{Im} F$ = const, and on the free surface $\mathrm{Re} F = 0$ in accordance with (4.2). Knowing the singularities of the function F in the canonical region of the complex variable ζ and the simplicity of boundary conditions permit to construct the following function

$$V_0 \frac{dz}{dw} \exp \frac{i\alpha z}{2l} = e^{F(\zeta)}$$

Multiplying both parts of the last equation by $dw/d\zeta = f(\zeta)$, we obtain a differential equation for determination of $z(\zeta)$

$$V_0 \frac{dz}{d\zeta} \exp \frac{i\alpha z}{2l} = e^{F(\zeta)} f(\zeta)$$

From this

$$z = \frac{2l}{i\alpha} \ln \left\{1 + \frac{i\alpha}{2lV_0} \exp\left[-\frac{i\alpha}{2l} z(\zeta_1)\right] \int_{\zeta_1}^{\zeta} e^{F(\zeta)} f(\zeta)\, d\zeta\right\} + z(\zeta_1)$$

The method described was used by the author for the solution of the problem of cavitating flow around a flat plate by a stream of a heavy liquid [6]. Let us now apply this method to the problem of a vortex in a finite mass of a heavy liquid. We will have here

$$V_0 \frac{dz}{dw} \exp \frac{i\alpha z}{2l} = \zeta e^{i\nu}, \qquad \frac{dw}{d\zeta} = \frac{\Gamma}{2\pi i} \frac{1}{\zeta}$$

From this

$$V_0 \frac{2l}{i\alpha} \exp \frac{i\alpha z}{2l} = \frac{\Gamma}{2\pi i} e^{i\nu}\zeta + C_1 + iC_2$$

Real constants Γ, ν, C_1 and C_2 are determined from conditions

$$z(i) = il, \qquad z(-i) = -il$$

In the final form

$$z = \frac{2l}{i\alpha}\left[\ln\left(1 + i\zeta \tanh\frac{\alpha}{2}\right) + \ln\cosh\frac{\alpha}{2}\right], \qquad \Gamma = 4\pi l V_0 \frac{1}{\alpha}\sinh\frac{\alpha}{2} \tag{4.3}$$

Expanding (4.3) in series with respect to α we find

$$= l\,[\zeta - {}^1/_4\, i\alpha(1+\zeta^2) - {}^1/_{12}\alpha^2(\zeta+\zeta^3) - \ldots], \quad \Gamma = 2\pi l V_0 (1 + {}^1/_{24}\alpha^2 + \ldots) \tag{4.4}$$

Comparing the last expression with Equations (1.10) and (1.11) we convince ourselves that Equations (4.3) solve the problem in the first approximation.

The author is thankful to M.I. Gurevich for useful suggestions and advice.

BIBLIOGRAPHY

1. Woronetz, K., L'influence de la pesanteur sur la forme du jet liquide. Compt.rend.Acad.Sci., № 3, 236, 1953.
2. Woronetz, K., L'influence des forces extérieures sur l'écoulement par les orifices. Publs.Inst.math.Acad.Serbe Sci., Vol.5, 1953.
3. Gurevich, M.I. and Pykhteev, G.N., Priblizhennoe reshenie zadachi ob istechenii tiazheloi ideal'noi neszhimaemoi zhidkosti iz-pod shchita (Approximate solution of the problem of flow of a heavy ideal incompressible fluid from under a baffle). PMTF № 2, 1960.
4. Kostychev, G.I., O postroenii krylovogo profilia po khordovoi diagramme reaspredeliniia skorosti ili davleniia (Construction of a wing profile from a chord diagram of velocity or pressure distribution). Tr.Kazansk.aviats.in-ta, Vol.38, 1958.
5. Kantorovich, L.V., O konformnom otobrazhenii (On conformal mapping). Matem.sb., Vol.40, № 3, 1933.
6. Kiselev, O.M., O kavitatzionnom obtekanii plastinki potokom tiazheloi zhidkosti (On cavitational flow of a stream of heavy liquid around a flat plate). Izv.vyssh.uchebn.zaved., Matematika, № 6, 1963.

Translated by B.D.